Differential equation

(1) B.Sc. part-2 paper IV

Homogeneous differential equation

Def:- A function $F(x,y)$ is said to be homogeneous of degree n if $F(Mx, My) = M^n F(x,y)$ for any non zero constant M

Such equation can be solved by the substitution $y = vx$ and so $\frac{dy}{dx} = v + x\frac{dv}{dx}$

problem (1) $x^2y\,dx - (x^3+y^3)dy = 0$

Soln:-

$$\frac{dy}{dx} = \frac{x^2y}{x^3+y^3} \quad \text{---(1)}$$

we put $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Hence from (1)

$$v + x\frac{dv}{dx} = \frac{x^2vx}{x^3+v^3x^3}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v}{1+v^3} - v = \frac{v-v-v^4}{1+v^3}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-v^4}{1+v^3}$$

$$\Rightarrow \frac{dx}{x} = -\frac{1+v^3}{v^4}dv$$

$$\Rightarrow \int\frac{dx}{x} = -\int v^{-4}dv - \int\frac{1}{v}dv$$

$$\Rightarrow \log x + \log K = \frac{-v^{-4+1}}{-4+1} - \log v$$

$$\Rightarrow \log Kx + \log\frac{y}{x} = \frac{v^{-3}}{3} = \frac{x^3}{3y^2} \; [v = \frac{x^3}{y^3}$$

$$\Rightarrow \log Ky = x^3/3y^2$$

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problem (2) solve $(x^2-y^2)\frac{dy}{dx}=2xy$

Soln:- $\frac{dy}{dx}=\frac{2xy}{x^2-y^2}$ — (1)

we put $y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

By (1) $v+x\frac{dv}{dx}=\frac{2xvx}{x^2-v^2x^2}$

$\Rightarrow x\frac{dv}{dx}=\frac{2v}{1-v^2}-v=\frac{2v-v+v^3}{1-v^2}$

$\Rightarrow \frac{dx}{x}=\frac{1-v^2}{v+v^3}dv=\frac{1+v^2-2v^2}{v(1+v^2)}dv$

$\Rightarrow \int\frac{dx}{x}=\int\frac{1}{v}dv-\int\frac{2v}{1+v^2}dv$

$\Rightarrow \log x=\log v-\log(1+v^2)+\log K$

$\Rightarrow \log x=\log\frac{vK}{1+v^2}$

$\Rightarrow x=\frac{yK}{x\left(\frac{x^2+y^2}{x^2}\right)} \Rightarrow x=\frac{xyK}{x^2+y^2}$

$\therefore x^2+y^2=Ky$

problem (3) $(x^2+y^2)\frac{dy}{dx}=2xy$

Soln:- $\frac{dy}{dx}=\frac{2xy}{x^2+y^2}$ we put $y=vx$

$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

$\therefore v+x\frac{dv}{dx}=\frac{2xvx}{x^2+v^2x^2}=\frac{2v}{1+v^2}$

$\Rightarrow x\frac{dv}{dx}=\frac{2v}{1+v^2}-v=\frac{2v-v-v^3}{1+v^2}=\frac{v-v^3}{1+v^2}$

$\Rightarrow \frac{dx}{x}=\frac{1+v^2}{v(1-v^2}dv$

$$= \frac{1+v^2}{v(1-v)(1+v)}$$

Let

$$\frac{1+v^2}{v(1-v)(1+v)} = \frac{A}{v} + \frac{B}{1-v} + \frac{C}{1+v}$$

$$\Rightarrow \frac{1+v^2}{v(1-v)(1+v)} = \frac{A(1-v)(1+v)+Bv(1+v)+Cv(1-v)}{v(1-v)(1+v)}$$

when $v=0$ then $A=1$

where $v=1$ then $B=1$

when $v=-1$ then $C=-1$

Now from (1)

$$\int \frac{dx}{x} = \int \frac{1}{v}\,dv + \int \frac{1}{1-v}\,dv - \int \frac{1}{1+v}\,dv$$

$$\Rightarrow \log x + \log K = \log v - \log(1-v) - \log(1+v)$$

$$\Rightarrow \log Kx = \log \frac{v}{1-v^2}$$

$$\Rightarrow Kx = \frac{y}{x\left(\frac{x^2-y^2}{x^2}\right)} \Rightarrow K(x^2-y^2) = y$$

problem (4) $(1+e^{x/y})dx + e^{x/y}(1-\frac{x}{y})dy = 0$

soln:- $\frac{dx}{dy} = \frac{-e^{x/y}(1-\frac{x}{y})}{1+e^{x/y}}$

we put $x = vy$ $\therefore \frac{dx}{dy} = v + y\frac{dv}{dy}$

$$\Rightarrow \frac{dx}{dy} = -\frac{e^v(1-v)}{1+e^v}$$

$$\Rightarrow y\frac{dv}{dy} = -\frac{e^v(1-v)}{1+e^v} - v$$

$$\Rightarrow y\frac{dv}{dy} = \frac{-e^v + ve^v - v - ve^v}{1+e^v}$$

$$\Rightarrow y\frac{dv}{dy} = \frac{-(e^v+v)}{(1+e^v)}$$

$$\Rightarrow \int \frac{dy}{y} = -\int \frac{1+e^v}{v+e^v} = \cancel{\log k + \log y}$$

$$\Rightarrow \log k + \log y = -\log(v+e^v)$$

$$\Rightarrow \log ky + \log \frac{y}{x} + e^{x/y} = 0$$

$$\Rightarrow ky\left(\frac{x + y e^{x/y}}{y}\right) = e^0$$

$$\Rightarrow ky\left(\frac{x + y e^{x/y}}{y}\right) = 1$$

$$x + y e^{x/y} = 1$$